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Integral-equation approach to the instability of two-dimensional sheared flow of inviscid fluid in a rotating system with variable Coriolis parameter

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Abstract. An integral-equation approach to the linear barotropic-instability problem is studied. A certain regularization process is used to construct an integral equation which is free of inconvenient singularities and is tractable by classical methods. Other advantages of the integral-equation approach over the conventional differential-equation approach based on the so-called Rayleigh–Kuo differential equation are studied.

1. Introduction

It is well known that in a two-dimensional sheared flow of inviscid fluid in a rotating system with a variable Coriolis parameter, a so-called barotropic instability can develop when $S'(y) = \beta - U''(y)$ vanishes somewhere in the flow field $U(y)$. Here $S(y)$ is the absolute vorticity of the basic flow field $U(y)\boldsymbol{i}_x$ which is the sum of the planetary vorticity β and the relative vorticity of the flow itself (a prime denotes a y-derivative with x eastward and y northward). The linear normal-mode equation underlying this instability is a second-order differential equation that possesses a singularity where the phase speed c of a wave matches the local flow speed U. The linear theory (Kuo 1973, Tung 1981) predicts that the wave is barotropically instable when its phase speed c lies within the band $U_c - \Delta U_1 < c < U_c + \Delta U_2$, where U_c is the critical flow speed and ΔU_1 and ΔU_2 are determined by $S'(y) < 0$. The modes with marginal $c (c = U_c - \Delta U_1$ or $c = U_c + \Delta U_2)$ are called neutral. Hence the idea is that only a narrow region of the flow field $U(y)$, where $S'(y) < 0$, is responsible for a resonant interaction between the flow field and the wave whereas the bulk of the basic flow carries the wave and is responsible for its dispersive properties. Thus the problem of barotropic instability is very analogous to that of the excitation of a monochromatic plasma wave by an electron beam (Nicholson 1983, Churilov 1989). Both processes are (apart from notation differences) described by similar equations, in which the absolute vorticity S (with opposite sign) and the electron-velocity distribution play equivalent roles.

Guided by this idea we present in this paper an integral-equation approach to the (linear) instability of a zonal sheared flow. Although the analysis presented in this

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paper applies also to the same stability problem without rotation (in which case we are dealing with the classical Rayleigh stability problem) we nevertheless include rotation for reasons of completeness remarking that it is always possible to reduce the problem to the classical Rayleigh problem by putting β equal to zero (see Drazin and Howard 1966).

We start in section 2 with a formulation of the barotropic-instability problem and derive the well known Rayleigh–Kuo equation for the normal-mode solutions, say the streamfunction perturbations. In section 3 we proceed by discussing an integro-differential equation that we derive for the vorticity perturbation and of which the integral operator turns out to be non-separable. Section 4 deals with the formal solution of the integro-differential equation. Section 5 demonstrates the Rayleigh–Kuo inflection-point theorem in that we establish that, if S' does not change sign in the flow, then the integral operator of our integro-differential equation is self-adjoint with respect to the proper scalar product. In section 6 we present the normal-mode theory for the vorticity perturbation and derive an integral equation for the eigenfunctions, which has the form of a multiplication operator perturbed by an integral operator. A method for solving this integral equation is then used to find the continuum eigenmodes. We furthermore give in that section a classification of the various eigenmodes of the Rayleigh–Kuo equation. Thus we are able to identify a new (implicit) form for the dispersion function governing the instability. Section 7 is devoted to the relationship between the conventional differential-equation approach to the barotropic-instability problem and the present integral-equation approach. In section 8 the initial-value problem for the barotropic-instability problem is evaluated. Special attention is paid to the asymptotic behaviour of the perturbation streamfunction for $t \to \infty$. The influence of the logarithmic singularity in the perturbation streamfunction in the critical layer on this long-time behaviour is particularly discussed thereby identifying a misunderstanding present in the literature. The disappearance of the continuous spectrum associated with the cuts due to this logarithmic singularity when the flow develops a sharp jump and becomes constant elsewhere (vortex sheet) is conjectured to be related to the mathematical phenomenon of what is called spectral concentration. In section 9 we employ our new dispersion function to study the barotropic instability and thus derive expressions for its growth rate in terms of the neutral solutions. A way of constructing a long-wave approximation similar to the one derived by Howard and Drazin (1964) but now for a bounded flow is presented in section 10. An important difference is that our long-wave approximation to the dispersion relation retains the logarithmic multi-valuedness contrary to the one of Howard and Drazin (1964). Section 11 finally deals with the construction of an exact (explicit) dispersion function by expanding the integral equation's kernel in some complete set of functions thus making its solution algebraic in nature. This exact dispersion function is in fact a determinantal equation that can be viewed upon as an alternative of the Fredholm determinant. Some special cases in which it can be solved exactly are considered.

2. The basic problem

Consider a basic steady two-dimensional flow of incompressible inviscid homogeneous fluid in a Cartesian x_* (eastward)–y_* (northward)-coordinate system (the subscript asterisks denote dimensional quantities; we shall soon drop them to use dimensionless quantities). The flow is assumed to be sheared in the y_*-direction and to be bounded by two rigid parallel planes, $y_* = y_{1*}$ and $y_* = y_{2*}$ either of which may be at infinity. Furthermore we assume that our system rotates with an angular speed Ω_{z*} perpendicular to the flow. It is also assumed that the curvature of the earth is negligible but the variations with latitude ϕ of the Coriolis parameter $f_* = 2\Omega_{z*} = 2\Omega_* \sin(\phi)$ is retained (Ω_* is the earth's angular speed) and furthermore that the meridional component Ω_{y*} of the earth's angular speed is neglected.

The flow is now governed by the following equation of motion:

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}p + \rho\boldsymbol{g} - 2\rho(\boldsymbol{\Omega}\times\boldsymbol{u}) \tag{2.1}$$

where ρ is the mass density, $\boldsymbol{u} = (u, v, w)$ is the fluid velocity, p is the pressure and $\boldsymbol{g}$ is the gravitational acceleration ($\boldsymbol{g} = -g\boldsymbol{i}_z$) and where the variables have been made dimensionless in the usual way with some velocity scale U of the basic flow $U_*(y_*)$ and some length scale L of the variations of the basic flow. The other symbols in (2.1) have their usual meaning. In (2.1) we also have neglected centrifugal forces.

Introducing the vorticity of the flow $\boldsymbol{\zeta} = \boldsymbol{\nabla}\times\boldsymbol{u}$ and upon taking the curl of (2.1) we obtain

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \boldsymbol{\omega}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p}{\rho^2} \tag{2.2}$$

where d/dt is the substantial derivative. $\boldsymbol{\omega}$ is the so-called absolute vorticity of the fluid as observed from an inertial non-rotating frame, that is

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times(\boldsymbol{u} + \boldsymbol{\Omega}\times\boldsymbol{r}) = \boldsymbol{\zeta} + 2\boldsymbol{\Omega}. \tag{2.3}$$

It is finally assumed that the flow is incompressible ($\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$) and geostrophic with no horizontal temperature gradients ($\partial/\partial z \equiv 0$), that the relative vorticity $\boldsymbol{\zeta}$ is much smaller than the planetary vorticity $2\boldsymbol{\Omega}$ and that the fluid is barotropic ($\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}p \equiv \boldsymbol{0}$), then for small Rossby number $R_0 = U/(fL)$ the vorticity equation (2.2) may be approximated by

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\zeta + f)\boldsymbol{i}_z = 0 \tag{2.4}$$

which denotes conservation of absolute vorticity (frozen-in vortex lines).

Equation (2.4) turns out to describe adequately well the synoptic behaviour of a meridionally sheared zonal flow of barotropic fluid at mid-latitudes. It will be the starting eqution for our subsequent considerations.

Since we presume that the flow is two-dimensional and solenoidal, the velocity field $\boldsymbol{u}$ may be expressed in terms of a streamfunction $\psi = \psi(x, y)$ defined by letting

$$u = -\frac{\partial\psi}{\partial y} \qquad v = \frac{\partial\psi}{\partial x} \tag{2.5}$$

from which it is easily verified that

$$\zeta = \boldsymbol{\nabla}^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}. \tag{2.6}$$

Substituting this into (2.4) yields

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \mathrm{J}(\psi, \nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = 0 \tag{2.7}$$

where the Jacobian of two functions

$$\mathrm{J}(A, B) = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x} \tag{2.8}$$

has been introduced and where we have expanded the Coriolis parameter f in a Taylor series about the latitude ϕ_0 as

$$f = f_0 + \beta y + \text{ (higher order terms)} \tag{2.9}$$

where

$$\beta = \left.\frac{\mathrm{d}f}{\mathrm{d}y}\right|_{\phi=\phi_0} \qquad (\phi = \phi_0 \Rightarrow y = 0). \tag{2.10}$$

β is henceforth assumed to be constant. The boundary conditions for (2.7) are

$$\psi(x, y = y_1) = 0 = \psi(x, y = y_2). \tag{2.11}$$

In order to linearize (2.7) we assume that the dependent variables u, v and ψ are a superposition of an atmospheric mean state with a sheared zonal flow $U(y)$ and a streamfunction $\Psi(y)$ and unsteady perturbations $u(x, y, t), v(x, y, t)$ and $\Psi(x, y, t)$ according to

$$\begin{aligned} u &= U(y) + u(x, y, t) = -\frac{\mathrm{d}\Psi}{\mathrm{d}y} + u(x, y, t) \\ v &= v(x, y, t) \\ \psi &= \Psi(y) + \psi(x, y, t). \end{aligned} \tag{2.12}$$

Then the vorticity equation (2.7) may be written in the following form:

$$\left\{\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right\}\nabla^2\psi + \frac{\mathrm{d}S}{\mathrm{d}y}\frac{\partial\psi}{\partial x} = 0 \tag{2.13}$$

where $S = S(y)$ is the absolute vorticity of the basic state, i.e.

$$S(y) = f(y) - \frac{\mathrm{d}U}{\mathrm{d}y} \qquad \frac{\mathrm{d}S}{\mathrm{d}y} = \beta - \frac{\mathrm{d}^2U}{\mathrm{d}y^2}. \tag{2.14}$$

Upon taking a harmonic x- and t-dependence according to

$$\psi(x, y, t) = \hat{\psi}(y; c, k)\exp\{\mathrm{i}k(x - ct)\} \tag{2.15}$$

(2.13) reduces to the well known Rayleigh–Kuo equation (Rayleigh 1880, Kuo 1949), i.e.

$$\{U(y) - c\}\left(\frac{\mathrm{d}^2\hat{\psi}}{\mathrm{d}y^2} - k^2\hat{\psi}\right) + S'(y)\hat{\psi} = 0. \tag{2.16a}$$

The prime denotes differentiation with respect to y. Equation (2.16*a*) is subjected to the boundary conditions

$$\hat{\psi}(y = y_1; c, k) = 0 = \hat{\psi}(y = y_2; c, k). \tag{2.16b}$$

k is the zonal wavenumber and c is the phase speed of the wave (2.15). If the flow is unbounded (which corresponds to the interpretation of y as a Mercator coordinate), it can happen that for some real c the streamfunction $\hat{\psi}$ oscillates in y at infinity without damping or amplification. In that case (2.16*b*) should be replaced by Sommerfeld's radiation condition. Equations (2.16*a*) and (2.16*b*) constitute what is known as the barotropic-instability problem. That is an eigenvalue problem for the phase speed c which may be complex, i.e.

$$c = c_\mathrm{r} + \mathrm{i}c_\mathrm{i} = \frac{\nu}{k} = \frac{\sigma + \mathrm{i}\gamma}{k} \tag{2.17}$$

where ν is the wave's angular frequency.

If $c_\mathrm{i} > 0$, the solution (2.15) of (2.16) is amplified (instability); if $c_\mathrm{i} < 0$, the solution is damped; and if $c_\mathrm{i} = 0$, the solution of (2.16) is said to be neutral. Since the problem is invariant under complex conjugation, a damped solution with $c_\mathrm{i} < 0$ implies the existence of an amplified conjugate solution with $c_\mathrm{i} > 0$ (and *vice versa*). Thus instability of a disturbance of a given wavenumber corresponds to complex c and stability to real c.

If the phase speed is real, so for neutral solutions, there is a singularity in (2.16*a*) at the critical level $y = y_\mathrm{c}$ where $U(y_\mathrm{c}) - c = 0$. Lindzen and Tung (1978) argued that the instability of the sheared flow is caused by wave over-reflection from this critical layer.

In what follows we shall restrict ourselves to flow profiles $U(y)$ that are continuous and also have continuous derivatives. Discontinuous profiles should be considered as limits of continuous ones.

Equation (2.16) constitutes a Sturm–Liouville equation for which the theory is well known and although there is a wealth of standard equations of mathematical physics one could revert to when (2.16) needs to be solved for various profiles $U(y)$, in practice it could be difficult to do so for some particular one. Moreover we have to deal with singularities there where the flow matches the phase speed c. These singularities are determined by the possibly multi-valued function.

$$y = y_\mathrm{c} = U^{-1}(c). \tag{2.18}$$

Nevertheless it is possible to obtain a necessary (but in general not sufficient) condition for instability. It was in fact Kuo (1949) who generalized the well known inflection-point theorem of Rayleigh (1880). It states that a necessary condition for the barotropic instability is that $S' = \beta - U''$ changes sign somewhere between y_1 and y_2.The region where $S' < 0$ can be shown to be the region of instability. Changing the sign of S', however, is not always a sufficient condition for the barotropic instability as can be shown by a counterexample with $U(y) = \sin(y)$ (see Howard and Drazin 1964). At the location of the singularity one of the two independent solutions of (2.16) remains finite whereas the other becomes logarithmically singular as a local Frobenius expansion reveals and furthermore undergoes a sudden change of phase. To avoiding this unphysical difficulty one could take into consideration the effect of viscosity which turns out to be necessary only in a narrow layer around the critical level at $y = y_c$ (Lin 1955, 1957). The higher-order equation then obtained is the well known Orr–Sommerfeld equation that reduces to its degenerate form (2.16) for zero viscosity (Kuo 1949) and can be studied by means of boundary-layer techniques. In this picture the nonlinear evolution of the barotropic instability results in lateral mixing and the production of eddy energy at the expense of kinetic energy of the zonal flow. In a self-consistent treatment of the problem it is found (Churilov 1989) that this nonlinear development of the instability results in a rearrangement of the absolute vorticity S of the basic state. The $S(y)$-profile is smoothed and homogenized thus leading to a 'plateau' formation. This is very analogous to what happens with a monochromatic Langmuir wave excited by an electron beam in a plasma. Both processes are described by similar equations in which the absolute vorticity (with opposite sign) and the electron-distribution function play equivalent roles. In the next section we shall consider an integral equation that is equivalent to the linearized Vlasov equation of plasma physics describing the linear evolution of the Langmuir wave mentioned hereinbefore.

3. An integro-differential equation for the perturbation vorticity

Consider once again the linearized vorticity equation (2.13) but now in the following form:

$$\frac{\partial s}{\partial t} + \mathrm{i}kU(y)s = -\mathrm{i}k\frac{\mathrm{d}S}{\mathrm{d}y}\psi = -\mathrm{i}k\frac{\mathrm{d}S}{\mathrm{d}y}\nabla^{-2}s \tag{3.1}$$

where we have introduced the perturbation vorticity s with

$$s(y,t;k) = \nabla^2\psi = \left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - k^2\right)\psi(y,t;k). \tag{3.2}$$

It may be noted that we already have Fourier transformed with respect to x. The expression on the right-hand side of (3.1) is actually an integral operator acting on the perturbation vorticity. We shall denote its kernel by $-Q(y \mid \eta)/k^2$, so that we can write for (3.1)

$$\left\{\frac{\partial}{\partial t} + \mathrm{i}kU(y)\right\} s(y,t;k) = \frac{-1}{\mathrm{i}k}\frac{\mathrm{d}s}{\mathrm{d}y}\int_{y_1}^{y_2} Q(y \mid \eta)s(\eta,t;k)\,\mathrm{d}\eta \tag{3.3}$$

where

$$\frac{\partial^2 Q}{\partial y^2} - k^2 Q = -k^2\delta(y-\eta) \tag{3.4}$$

in which δ is Dirac's function and

$$Q(y = y_1 \mid \eta) = 0 = Q(y = y_2 \mid \eta). \tag{3.5}$$

The boundary conditions to be superimposed on s are taken to be

$$s(y = y_1, t; k) = 0 = s(y = y_2, t; k) \tag{3.6}$$

which, in effect, is equivalent to (2.16*b*) since whenever some eigenfunction $\hat{\psi}$ solves (2.16*b*), then

$$\hat{s} = \nabla^2\hat{\psi} = -\frac{S'(y)}{U(y)-c}\hat{\psi} = 0 \qquad \text{for } y = y_1, y_2. \tag{3.7}$$

Equation (3.3) is interesting in itself, as it poses the barotropic-instability problem under consideration in a physically elucidating way. The left-hand side represents the free oscillations of the individual vortex sheets as if they constitute an infinite set of linear (latitudinal) oscillators with continuously distributed eigenfrequencies $kU(y)$, whereas the right-hand side, i.e. the integral operator, describes the coupling. The left-hand side of (3.3) is called the free-streaming term, whereas the right-hand side represents the interaction.

Equation (3.3) is analogous to integro-differential equations describing linear oscillations of a hot (Nicholson 1983) and a cold plasma (Sedláček 1971b) as well as to an integro-differential equation describing anisotropic neutron transport (Sattinger 1966).

4. Formal solution of the problem

To obtain a formal solution of (3.3), we rewrite this integro-differential equation as a Cauchy problem

$$\frac{-1}{\mathrm{i}k}\frac{\partial s}{\partial t} = \mathcal{B}s \tag{4.1}$$

where $\mathcal{B}$ is the bounded operator

$$\mathcal{B}s(y,t;k) = U(y)s(y,t;k) - \frac{1}{k^2}\frac{\mathrm{d}S}{\mathrm{d}y}\int_{y_1}^{y_2} Q(y \mid \eta)s(\eta,t;k)\,\mathrm{d}\eta. \tag{4.2}$$

Since $\mathcal{B}$ is bounded, the spectrum of $\mathcal{B}$ is compact and Howard's semi-circle theorem (Howard 1961) implies that the point spectrum of $\mathcal{B}$ lies within the upper semi-circle with centre $(U_{\min} + U_{\max})/2$ and radius $(U_{\max} - U_{\min} + \beta/k^2)/2$. Assuming tacitly and henceforth (without loss of generality) that $k > 0$, the formal solution of (3.3) is now given by

$$s(y,t;k) = \mathcal{K}s(y,0;k) \tag{4.3}$$

where

$$\mathcal{K} = \exp(-\mathrm{i}kt\mathcal{B}). \tag{4.4}$$

Laplace transforming (4.1) yields the following alternative representation of the operator:

$$s(y,t;k) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\exp(-\mathrm{i}ktc)(c-\mathcal{B})^{-1}s(y,0;k)\,\mathrm{d}c \tag{4.5}$$

where Γ is a curve encircling the spectrum of $\mathcal{B}$.

We now give an alternative expression for the resolvent $(\mathcal{B}-c)^{-1}$ in terms of the Green function of the Rayleigh–Kuo equation. For that we observe that $\hat{s} := (\mathcal{B}-c)^{-1}\hat{\phi}$ with $\hat{\phi}(y=y_1) = 0 = \hat{\phi}(y=y_2)$ is a solution of the equation

$$\begin{aligned}\nabla^2\hat{\psi} + \frac{S'(y)}{U(y)-c}\hat{\psi} &= \frac{\mathrm{d}^2\hat{\psi}}{\mathrm{d}y^2} - \left\{k^2 - \frac{S'(y)}{U(y)-c}\right\}\hat{\psi} \\ &= \nabla^2\left\{\frac{\hat{\phi}}{S'(y)}\right\} = \frac{\mathrm{d}^2}{\mathrm{d}y^2}\left\{\frac{\hat{\phi}}{S'(y)}\right\} - \frac{k^2}{S'(y)}\hat{\phi}\end{aligned} \tag{4.6}$$

with

$$\hat{\psi} = \frac{U(y)-c}{S'(y)}\hat{s}. \tag{4.7}$$

Thus if $G(y \mid \eta)$ is the Rayleigh–Kuo equation's Green function, then

$$\begin{aligned}\hat{s}(y;c,k) &= (\mathcal{B}-c)^{-1}\hat{\phi}(y;c,k) \\ &= \frac{S'(y)}{U(y)-c}\int_{y_1}^{y_2} G(y \mid \eta)\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\left\{\frac{\hat{\phi}(\eta;c,k)}{S'(\eta)}\right\} - \frac{k^2}{S'(\eta)}\hat{\phi}(\eta;c,k)\right]\mathrm{d}\eta.\end{aligned} \tag{4.8}$$

If $\hat{\psi}_1$ and $\hat{\psi}_2$ are two independent solutions of the Rayleigh–Kuo equation with $\hat{\psi}_1(y = y_1) = 0 = \hat{\psi}_2(y = y_2)$, then

$$G(y \mid \eta) = \frac{\hat{\psi}_1(y_<)\hat{\psi}_2(y_>)}{\mu(c,k)} \tag{4.9}$$

where

$$\mu(c,k) = \mathrm{W}[\hat{\psi}_1, \hat{\psi}_2] = \hat{\psi}_1\frac{\mathrm{d}\hat{\psi}_2}{\mathrm{d}y} - \hat{\psi}_2\frac{\mathrm{d}\hat{\psi}_1}{\mathrm{d}y}. \tag{4.10}$$

W denotes the Wronskian which in the present case does not depend upon y. Thus it becomes clear that the poles of $(\mathcal{B}-c)^{-1}$ correspond to the zeroes of $\mu(c,k)$.

In general the formal solution (4.3) of the intial-value problem (4.1) is of less practical use because finding the resolvent $(\mathcal{B}-c)^{-1}$ is tantamount to solving the Rayleigh–Kuo equation. However, for example, plane Poiseuille flow with

$$U(y) = \frac{\mathcal{B}}{2}(y-y_0)^2 + U_0 \tag{4.11}$$

where y_0 and U_0 are arbitrary constants the situation becomes very simple. In this case $S'(y) \equiv 0$ and $\mathcal{B}$ according to (4.2) reduces to a multiplication operator,

$$\mathcal{B} = U(y). \tag{4.12}$$

Now (4.3) and (4.4) yield

$$s(y,t;k) = \exp\{-\mathrm{i}kU(y)t\}s(y,0;k) \tag{4.13}$$

denoting that the perturbation vorticity at point y at time t is obtained from that at time 0 merely by multiplication with the phase factor $\exp\{-\mathrm{i}kU(y)t\}$. In this case $\mathcal{B}$ does not possess a point spectrum but only a continuous spectrum. An eigensolution can easily be found for each k and c in the range $U_{\mathrm{min}} \leq c \leq U_{\mathrm{max}}$.

5. Self-adjointness

Consider again the integral equation (4.1) for $s = \hat{s}\exp(-\mathrm{i}\omega t)$, i.e.

$$
\begin{aligned}
c\hat{s}(y;c,k) = \mathcal{B}\hat{s}(y;c,k) &= U(y)\hat{s}(y;c,k) - \frac{1}{k^2}\frac{\mathrm{d}S}{\mathrm{d}y}\int_{y_1}^{y_2} Q(y \mid \eta)\hat{s}(\eta;c,k)\,\mathrm{d}\eta \\
&= (\mathcal{F}+\mathcal{I})\hat{s}(y;c,k)
\end{aligned}
\tag{5.1}
$$

where we have split the operator $\mathcal{B}$ into an operator $\mathcal{F} = U(y)$ that is called the free-streaming operator for it corresponds to the free, mutually independent oscillations of the fluid and an opertor $\mathcal{I} = S'(y)\nabla^{-2}$ that accounts for the interaction. Clearly the operator $\mathcal{B}$ is not Hermitian in the scalar product

$$
(f,g) = \int_{y_1}^{y_2} f^*(y)g(y)\,\mathrm{d}y. \tag{5.2}
$$

However, if we introduce the weight function $W(y) = 1/S'(y)$ and require that $W(y)$ is continuous and does not change sign in (y_1, y_2), then with

$$
\langle f,g\rangle = (f, Wg) \tag{5.3}
$$

$\mathcal{B}$ becomes Hermitian for we have

$$
\begin{aligned}
\langle f,\mathcal{B}g\rangle &= (f, W(Ug+\mathcal{I}g)) = (f, WUg) + (f, W\mathcal{I}g) \\
&= (WUf, g) + (W\mathcal{I}f, g) = \langle \mathcal{B}f, g\rangle
\end{aligned}
\tag{5.4}
$$

where we have used the symmetry property of the Green function. This shows that if $S'(y)$ is continuous and does not change sign in (y_1, y_2), then all the eigenvalues c of $\mathcal{B}$ are real and consequently the flow is stable. Thus we have, in fact, rederived the Rayleigh–Kuo inflection-point theorem. Moreover the foregoing implies that the eigenfunctions $\hat{s}_n(y;c_n,k)$ for the eigenvalues c_n are orthogonal. This can be used to obtain the following expansion of $s(y,t;k)$ for a prescribed initial condition $s(y,0;k)$:

$$
s(y,t;k) = \sum a_n(t)\hat{s}_n(y;c_n,k) + \int F(t;c,k)\hat{s}(y,t;c)\,\mathrm{d}c \tag{5.5}
$$

provided that the set of eigenfunctions is complete. The integral in (5.5) is to be taken over the continuous spectrum of $\mathcal{B}$. Thus a convenient consequence of the Hermiticity of $\mathcal{B}$ is that the adjoint eigenfunctions need not be introduced in the stable case.

If $S'(y)$ changes sign somewhere in the flow between y_2 and y_2, then we also have to consider, along with our fundamental equation (5.1), an adjoint equation which is taken to be

$$
c\bar{\hat{s}}(y;c,k) = U(y)\bar{\hat{s}}(y;c,k) - \frac{1}{k^2}\int_{y_1}^{y_2}\frac{\mathrm{d}S}{\mathrm{d}y}Q(y \mid \eta)\bar{\hat{s}}(\eta;c,k)\,\mathrm{d}\eta. \tag{5.6}
$$

Now, again it is straightforward to show that the eigenfunctions $\bar{\hat{s}}(y;c,k)$ and $\hat{s}(y;c',k)$ are orthogonal for $c \neq c'$ with respect to the scalar product (5.2).

6. Normal-mode analysis

The operator $\mathcal{B}$ in (5.1) consists, as we have seen in the preceding section, of an operator of multiplication by $U(y)$, perturbed by an integral operator $\mathcal{I}$. Eigenvalue problems for such type of operators have been studied previously for the case of oscillations in hot and cold plasmas (van Kampen 1955, Case 1959, Sedláček 1971b) and for the case of anisotropic neutron transport (Sattinger 1966a). Moreover Rosencrans and Sattinger (1966) studied a similar operator that occurs in the theory of hydrodynamic stability for two-dimensional parallel flow of an incompressible fluid.

In the present paper we follow a method of solving (5.1) presented by Sattinger (1966a,b) and Sedláček (1971b). For that we assume that the integral operator $\mathcal{I}$ in (5.1) and the multiplication operator $U(y)$ have identical continuous spectra; the integral operator's only effect being the creation of new point eigenvalues. Thus we conjecture the solution of (5.1) to be of the form of a delta function (i.e. the eigenfunction of the multiplication operator) multiplied by some function $D(c,k)$ plus a perturbation term (due to the integral operator), the overall form of which is given by

$$\hat{s}(y;c,k) = D(c,k)\delta\{U(y)-c\} - P\frac{S'(y)}{U(y)-c}\hat{\phi}(y;c,k). \tag{6.1}$$

Here P signifies that the principal value integral is to be taken when integrating the expression for $\hat{s}$ with respect to y.

The form of the normal-mode solutions (6.1) has in fact the same structure as the van Kampen eigenmodes of hot uniform plasma oscillations (van Kampen 1955). Equation (6.1) says that for any wavenumber k, there are an infinite number of normal modes, one for each value of c. Furthermore these normal modes are neither damped nor amplified, but exist for all time with real phase speed c. We also note that (6.1) seems to violate the linearization procedure we have employed in (2.12) and (2.13) because $\hat{s} \to \infty$ for $c \to U(y)$ due to the delta function. Nevertheless the modes (6.1) are of importance because of the possibility of creating a physically and mathematically acceptable perturbation by adding up many such modes.

We now substitute (6.1) into (5.1). This is legitimate since the kernel $Q(y \mid \eta)$ is continuous in y and η so that its product with a delta function is well defined. We thus find

$$\hat{\phi}(y;c,k) = -\frac{D(c,k)}{k^2|U'(y_c)|}Q(y \mid y_c) + \frac{1}{k^2}P\int_{y_1}^{y_2}\frac{S'(y)}{U(y)-c}Q(y \mid \eta)\hat{\phi}(\eta;c,k)\,\mathrm{d}\eta. \tag{6.2}$$

Specifying the function D by the assumption

$$\frac{D(c,k)}{|U'(y_c)|} = C + P\int_{y_1}^{y_2}\frac{S'(y)}{U(y)-c}\hat{\phi}(y;c,k)\,\mathrm{d}y \tag{6.3}$$

where C is an arbitrary real constant, the function $\hat{\phi}(y;c,k)$ turns out to be defned by the following inhomogeneous Fredholm equation of the second kind:

$$\hat{\phi}(y;c,k) = -\frac{C}{k^2}Q(y \mid y_c) + \frac{1}{k^2}\int_{y_1}^{y_2}H(y \mid \eta)\hat{\phi}(\eta;c,k)\,\mathrm{d}\eta \tag{6.4}$$

where

$$H(y \mid \eta) = \frac{Q(y \mid \eta) - Q(y \mid y_c)}{U(y)-c}S'(\eta). \tag{6.5}$$

The principal-value sign is now superfluous since the kernel H is finite for $y = y_c$ where $U(y_c) = c$. We have thus reduced the problem of solving the singular integral equation (5.1) to one of solving a non-singular integral equation. This must be accepted as a simplification because efficient approximative procedures exist for solving such integral equations. One such procedure will be utilized in section 10 to obtain a long-wave approximation to the dispersion function determining the discrete eigenvalues of $\mathcal{B}$ for a bounded flow. The real constant C is determined by

$$C = \int_{y_1}^{y_2} \hat{s}(y; c, k)\, \mathrm{d}y. \tag{6.6}$$

Henceforth we assume the perturbation vorticity $\hat{s}$ to be normalized such that $C = -k^2$ presuming that $\int_{y_1}^{y_2} \hat{s}\, \mathrm{d}y \neq 0$.

There remains the question as to the existence and uniqueness of a solution of (6.4). The answer is provided by Fredholm's alternative theorem (Hildebrand 1965) from which we deduce that for a solution of (6.4) to be unique the corresponding homogeneous integral equation (obtained when C is put equal to zero in (6.4)) must not have a non-trivial solution. Or stated otherwise, the vorticity equation must not possess any solution with

$$\int_{y_1}^{y_2} \hat{s}(y; c, k)\, \mathrm{d}y = 0. \tag{6.7}$$

If, however, the corresponding homogeneous integral equation does have a non-trivial solution, the existence of the solution of (6.4) is guaranteed if, and only if, the inhomogeneous term $Q(y \mid y_c)$ is orthogonal to every solution of the adjoint homogeneous equation. If this condition is not fulfilled, then C must be put equal to zero implying (6.7).

Following Case (1959) we now can classify the eigenvalues and eigenfunctions of (5.1) in four groups.

Class I. c is real and $S'(y_c) \neq 0$. The normalization (6.6) shows that

$$\hat{s}(y; c, k) = D(c, k)\delta\{U(y) - c\} - P\frac{S'(y)}{U(y) - c}\hat{\phi}(y; c, k) \tag{6.8}$$

where

$$\frac{D(c, k)}{|U'(y_c)|} = -k^2 + P\int_{y_1}^{y_2} \frac{S'(y)}{U(y) - c}\hat{\phi}(y; c, k)\, \mathrm{d}y. \tag{6.9}$$

Class II. c is real and $S'(y_c) = 0$. The solutions are again given by (6.8) and (6.9). The principal-value sign is not necessary now.

Class III. c is real, $S'(y_c) = 0$ and $D(c, k) = 0$. This can only occur for discrete eigenvalues c_n and the solutions are

$$\hat{s}_n(y; c_n, k) = -\frac{S'(y)}{U(y) - c_n}\hat{\phi}_n(y; c_n, k). \tag{6.10}$$

Class IV. c is complex. The solutions are given by

$$\hat{s}_n(y; c_n, k) = -\frac{S'(y)}{U(y) - c_n}\hat{\phi}_n(y; c_n, k). \tag{6.11}$$

where due to the normalization (6.6) the phase speed c again is discrete with $D(c, k) = 0$.

Thus we find that the spectrum of $\mathcal{B}$ is continuous provided that the following do not hold simultaneously

$$S'(y_c) = 0 \tag{6.12a}$$

and

$$\frac{D(c,k)}{|U'(y_c)|} = -k^2 + \int_{y_1}^{y_2} \frac{S'(y)}{U(y)-c}\hat{\phi}(y;c,k)\,\mathrm{d}y = 0. \tag{6.12b}$$

Note that we have tacitly assumed that $U'(y_c) \neq 0$. Moreover it is now clear that the point spectrum of $\mathcal{B}$, that is the set of discrete normal-mode solutions of (5.1), is determined by

$$\frac{D(c,k)}{|U'(y_c)|} = 0 \tag{6.13}$$

for which reason we call $D(c,k)/|U'(y_c)|$ the dispersion function. Setting $D = 0$ does reduce the inhomogeneous non-singular Fredholm equation (6.4) to

$$\hat{\phi}(y;c,k) = \frac{1}{k^2}\int_{y_1}^{y_2} \frac{S'(\eta)}{U(\eta)-c} Q(u \mid \eta)\hat{\phi}(\eta;c,k)\,\mathrm{d}\eta \tag{6.14}$$

which is a homogeneous singular Fredholm integral equation. Consistent with $D(c,k) = 0$, this equation may have solutions for certain isolated values of c. In contrast to the standard Fredholm theory the eigenvalue parameter c is contained in the equation in an unusual manner. Nevertheless the Fredholm theory may apply and the Fredholm determinant then gives the dispersion function again.

7. On the relationship between a differential- and an integral-equation approach

The ansatz (6.1) can be straightforwardly related to the Rayleigh–Kuo equation (2.16a). For that we note that (2.16a) has two classes of solutions.

Class I. Discrete solutions that satisfy

$$\frac{\mathrm{d}^2\hat{\psi}}{\mathrm{d}y^2} - \left\{k^2 - \frac{S'(y)}{U(y)-c}\right\}\hat{\psi} = 0 \tag{7.1}$$

and the boundary conditions.

Class II. Solutions which satisfy the boundary conditions and

$$\frac{\mathrm{d}^2\hat{\psi}}{\mathrm{d}y^2} - \left\{k^2 - \frac{S'(y)}{U(y)-c}\right\}\hat{\psi} = \lambda(c,k)\delta\{U(y)-c\} \tag{7.2}$$

where λ is an arbitrary constant.

In the latter case we thus have a continuum of normal modes for c running between $U_{\min}$ and $U_{\max}$. Combining (7.2) with (3.2) gives

$$\hat{s}(y;c,k) + P\frac{S'(y)}{U(y)-c}\hat{\psi}(y;c,k) = \lambda(c,k)\delta\{U(y)-c\} \tag{7.3}$$

or

$$\hat{s}(y;c,k) = \lambda(c,k)\delta\{U(y)-c\} - P\frac{S'(y)}{U(y)-c}\hat{\psi}(y;c,k). \tag{7.4}$$

We thus have demonstrated that

$$D(c,k) = \lambda(c,k) \tag{7.5}$$

and that

$$\hat{\phi}(y;c,k) = \hat{\psi}(y;c,k) \tag{7.6}$$

where $\hat{\psi}(y;c,k)$ solves (7.2). In retrospect we now see that we have, in fact, regularized the integral equation (6.14) by making use of the arbitrariness of the jump in the derivative of the perturbation streamfunction at the critical layer. The solution of (7.2) is

$$\begin{aligned}\hat{\psi}(y;c,k) &= \frac{\lambda(c,k)}{|U'(y_c)|}G(y \mid y_c)\\ &= \frac{\lambda(c,k)}{\mu(c,k)|U'(y_c)|}\hat{\psi}_1(y_<;c,k)\hat{\psi}_2(y_>;c,k)\end{aligned} \tag{7.7}$$

where we have used (4.10) with $\eta = y_c$. Since the zeroes of $\lambda(c,k)$ and $\mu(c,k)$ coincide, this becomes

$$\hat{\psi}(y;c,k) = \frac{A}{|U'(y_c)|}\hat{\psi}_1(y_<;c,k)\hat{\psi}_2(y_>;c,k) \tag{7.8}$$

where A is a constant that is determined by the previously introduced normalization $\int_{y_1}^{y_2} \hat{s}\,\mathrm{d}y = -k^2$ (see (6.6)).

The integral equation (5.1) can also, after some manipulations, be written as follows

$$\begin{aligned}\{U(y)-c\}\hat{s}(y;c,k) &= \frac{1}{k^2}\int_{y_1}^{y_2}\{Q_1(\eta)Q_2(y) - Q_1(y)Q_2(\eta)\}\hat{s}(\eta;c,k)\,\mathrm{d}\eta\\ &\quad + \frac{1}{k^2}\frac{\mathrm{d}S}{\mathrm{d}y}Q_1(y)\int_{y_1}^{y_2} Q_2(\eta)\hat{s}(\eta;c,k)\,\mathrm{d}\eta\end{aligned} \tag{7.9}$$

where we have employed the notation

$$Q(y \mid \eta) = Q_1(y)Q_2(\eta) \tag{7.10}$$

for the kernel Q. It is easily shown that irrespective of the extension of the basic flow

$$Q_1(\eta)Q_2(y) - Q_1(y)Q_2(\eta) = k\sinh\{k(\eta-y)\}. \tag{7.11}$$

With this and the normalization

$$\int_{y_1}^{y_2} Q_2(y)\hat{s}(y;c,k)\,\mathrm{d}y = k^2 \tag{7.12}$$

(7.9) becomes

$$\hat{s}(y;c,k) = \frac{1}{k}\frac{S'(y)}{U(y)-c}\int_{y_1}^{y} \sinh\{k(\eta-y)\}\hat{s}(\eta;c,k)\,\mathrm{d}\eta + \frac{S'(y)}{U(y)-c}Q_1(y) \tag{7.13}$$

which is a Volterra integral equation with a constraint (7.12). With standard procedures this Volterra equation can be converted into the following differential equation for the perturbation vorticity $\hat{s}$:

$$\frac{\mathrm{d}^2\hat{s}}{\mathrm{d}y^2} - 2\frac{P'(y)}{P(y)}\frac{\mathrm{d}\hat{s}}{\mathrm{d}y} - \left[k^2 - P(y) + \frac{P''(y)}{P(y)} - 2\left\{\frac{P'(y)}{P(y)}\right\}^2\right]\hat{s} = 0 \tag{7.14}$$

where

$$P(y) = \frac{S'(y)}{U(y)-c}. \tag{7.15}$$

Taking

$$\hat{r} = \frac{\hat{s}}{S'(y)} \tag{7.16}$$

gives

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\{U(y)-c\}^2\frac{\mathrm{d}\hat{r}}{\mathrm{d}y}\right] + k^2\{U(y)-c\}\left[\frac{\beta}{k^2} - \{U(y)-c\}\right]\hat{r} = 0 \tag{7.17}$$

which is in fact the equation utilized by Howard and Drazin (1964) to obtain a long-wave approximation to the stability characteristics of unbounded flows.

8. The initial-value treatment

Instead of the normal-mode approach we pursued in section 6 to describe the barotropic-instability problem, we would now like to focus on an initial-value treatment. For that we follow Case (1960) in that we consider the perturbation vorticity equation (2.13) and take the Fourier transform with respect to x and Laplace transform with respect to t. Thus, let

$$\begin{aligned}\hat{\psi}(y;c,k) &= \int_0^{\infty} \mathrm{e}^{\mathrm{i}kct}\,\mathrm{d}t \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}kx}\psi(x,y,t)\,\mathrm{d}x \\ \psi(x,y,t) &= \frac{k}{(2\pi)^2}\int_{-\infty+\mathrm{i}\Gamma}^{+\infty+\mathrm{i}\Gamma} \mathrm{e}^{-\mathrm{i}ktc}\,\mathrm{d}c \int_{-\infty}^{+\infty} \mathrm{e}^{+\mathrm{i}xk}\hat{\psi}(y;c,k)\,\mathrm{d}k.\end{aligned} \tag{8.1}$$

Equation (2.13) then becomes

$$\{U(y)-c\}\left[\frac{\mathrm{d}^2\hat{\psi}}{\mathrm{d}y^2}-\left\{k^2-\frac{S'(y)}{U(y)-c}\right\}\hat{\psi}\right]=\frac{\Psi_0(y;k)}{\mathrm{i}k} \tag{8.2}$$

where

$$\Psi_0(y;k)=\frac{\mathrm{d}^2\psi(y,t=0;k)}{\mathrm{d}y^2}-k^2\psi(y,t=0;k). \tag{8.3}$$

The solution of (8.2) can be found with the Green function introduced in section 4 (see (4.10)) and is given by

$$\hat{\psi}(y;c,k)=\frac{1}{\mathrm{i}k}\int_{y_1}^{y_2}G(y\mid\eta)\frac{\Psi_0(\eta;k)}{U(y)-c}\,\mathrm{d}\eta. \tag{8.4}$$

Let us for a moment dwell upon the Green function $G(y|\eta)$ that is given by (4.9). The general solution of the Rayleigh–Kuo equation is

$$\hat{\psi}(y;c,k)=A\hat{\phi}_1(y;c,k)+B\hat{\phi}_2(y;c,k) \tag{8.5}$$

where $\hat{\phi}_1$ and $\hat{\phi}_2$ are two independent solutions. The boundary conditions demand that

$$\begin{aligned}A\hat{\phi}_1(y_1;c,k)+B\hat{\phi}_2(y_1;c,k)&=0\\ A\hat{\phi}_1(y_2;c,k)+B\hat{\phi}_2(y_2;c,k)&=0\end{aligned} \tag{8.6}$$

which has then, and only then, non-trivial solutions for the constants A and B if

$$\nu(c,k)=\hat{\phi}_1(y_1;c,k)\hat{\phi}_2(y_2;c,k)-\hat{\phi}_1(y_2;c,k)\hat{\phi}_2(y_1;c,k)=0. \tag{8.7}$$

The two independent solutions $\hat{\psi}_1$ and $\hat{\psi}_2$ that constitute the Green function (4.9) are given by

$$\begin{aligned}\hat{\psi}_1(y;c,k)&=\hat{\phi}_2(y_1;c,k)\hat{\phi}_1(y;c,k)-\hat{\phi}_1(y_1;c,k)\hat{\phi}_2(y;c,k)\\ \hat{\psi}_2(y;c,k)&=\hat{\phi}_2(y_2;c,k)\hat{\phi}_1(y;c,k)-\hat{\phi}_1(y_2;c,k)\hat{\phi}_2(y;c,k).\end{aligned} \tag{8.8}$$

The Wronskian of $\hat{\psi}_1$ and $\hat{\psi}_2$ is thus given by

$$\mathrm{W}[\hat{\psi}_1,\hat{\psi}_2]=\mu(c,k)=\nu(c,k)\mathrm{W}[\hat{\phi}_1,\hat{\phi}_2] \tag{8.9}$$

which, if we choose $\mathrm{W}[\hat{\phi}_1,\hat{\phi}_2]=1$, yields $\mu=\nu$.

To make any further progress possible we consider a very simple initial condition, that is

$$\Psi_0(y;k)=-k^2\delta(y-\xi). \tag{8.10}$$

This means that $\psi(y, t = 0; k)$ is given by the Green function $Q(y \mid \xi)$. Now (8.4) becomes

$$\hat{\psi}(y; c, k) = \mathrm{i}k \frac{G(y \mid \xi)}{U(\xi) - c} = \mathrm{i}k \frac{\hat{\psi}_1(y_<)\hat{\psi}_2(y_>)}{\mu(c, k)\{U(\xi) - c\}}. \tag{8.11}$$

The Laplace inversion of (8.11) yields

$$\psi(y, t; k) = \mathrm{i}\frac{k^2}{\pi^2} \int_{-\infty+\mathrm{i}\Gamma}^{+\infty+\mathrm{i}\Gamma} \frac{\hat{\psi}_1(y_<)\hat{\psi}_2(y_>)}{\mu(c, k)\{U(\xi) - c\}} \mathrm{e}^{-\mathrm{i}ktc} \, \mathrm{d}c \tag{8.12}$$

where Γ is to be taken such that the integration runs above all the singularities of the integrand. If we are interested in the asymptotic behaviour of $\psi(y, t; k)$ for $t \to \infty$ we may use standard Tauberian theorems (van der Pol and Bremmer 1950) for the Laplace transform. For that we have to deform the integration path in (8.12) downward thereby passing the singularities of the integrand. These are given by $c = U(\xi)$ and by $\mu(c, k) = 0$. To investigate the nature of these singularities, we construct a local Frobenius expansion (Ince 1956) for the functions $\hat{\phi}_1$ and $\hat{\phi}_2$

$$\begin{aligned} \hat{\phi}_1(y; c, k) &= (y - y_c)\left\{1 - \frac{S'(y_c)}{2U'(y_c)}(y - y_c) + \cdots\right\} \\ \hat{\phi}_2(y; c, k) &= \frac{S'(y_c)}{U'(y_c)}\hat{\phi}_1(y; c, k)\ln(y - y_c) - 1 + a_1(y - y_c) + \cdots \end{aligned} \tag{8.13}$$

provided that $U'(y_c) \neq 0$, where y_c is given the multi-valued inverse function $y_c = U^{-1}(c)$. Inserting (8.13) into the dispersion function (8.7) shows that ν and through (8.9) also μ, are multi-valued too the character of which is determined by a logarithmic term of the form

$$\ln\left\{\frac{y_1 - U^{-1}(c)}{y_2 - U^{-1}(c)}\right\}. \tag{8.14}$$

Thus to make the dispersion function single-valued, we have to cut the complex c-plane. If $U^{-1}(c)$ happens to be single-valued, these cuts are identical with the interval of the continuous spectrum of the Rayleigh–Kuo equation and in that case are referred to as spectral cuts (Sedláček 1971a). However, the multi-valuedness of $U^{-1}(c)$ could imply that these spectral cuts have to be extended, for example, to a cut along the real c-axis with $c_r \leq U_{\max}$.

Adam also considered the initial-value problem for a sheared flow (case IV′ of Adam (1984, 1986)). Although he does not take rotation into account, the mathematical structure of his equations is identical with ours. Moreover taking $\beta = 0$ in our treatment does reduce our problem to the one he considers. Adam (1984, 1986) states that the dispersion relation governing the normal modes of the dependent variable is single-valued in the complex c-plane and that this fact makes this case (case IV′ of (Adam 1984, 1986)) significantly different from some other problems he considers. As we have demonstrated hereinbefore this is incorrect. Although the variable he considers does not become infinite at $y = y_c$, its derivatives do and the c-plane has to be cut due to this logarithmic singularity, as it turns out to be, in order to make the dispersion relationship single-valued.

Returning to the asymptotic behaviour of the streamfunction perturbation $\psi(y,t;k)$, according to (8.12) it may be shown that the Laplace inversion gives rise to discrete normal modes due to zeroes of the dispersion function and due to the simple pole $1/(U-c)$ whereas the continuum due to the cuts that result from the logarithm and the multi-valued function $U^{-1}(c)$ give rise to something that vanishes as $1/t$ due to phase mixing of the continuum spectrum in physical space. Apart from the different asymptotic behaviour of the discrete normal modes and the continuous spectrum (exponential *versus* algebraic), there is another mathematically as well as physically important difference between them. Apart from the so-called free-streaming term $\exp\{-\mathrm{i}kU(y)t\}$, the various exponential discrete normal modes have frequencies that do not depend upon position and thus they represent the only modes that deserve to be called collective.

Contrary to the multi-valuedness introduced by $U^{-1}(c)$, the multi-valuedness of the dispersion function due to the logarithm is inevitable. Howard and Drazin (1964) derived an approximate dispersion relationship for long-wave solutions of the Rayleigh–Kuo equation. Their expressions, however, do not reveal any logarithmic term although it should be present as it is for $k=0$. The reason for this is probably that their dispersion relationship is in effect an approximation for sharp jumps in the basic flow $U(y)$. However, there are some quite significant differences between continuous and discontinuous flows. As we have argued heretofore, for a continuous flow there is a continuous spectrum the contributions of which are algebraically damped waves with frequencies that depend upon position. For a discontinuous flow, like, for example the vortex sheet, the spectrum is purely discrete. The whole fluid oscillates collectively (it there turns out that there are only two eigenmodes with conjugate eigenvalues for the vortex sheet). The question now arises as to what happens in the limiting process of a continuous profilt to make it discontinuous. We conjecture that the answer is provided by the mathematical phenomenon of spectral concentration (Titchmarsh 1951, Dolph 1961, Friedrichs 1966). An arbitrarily small perturbation may totally alter the nature of the spectrum, destroying the discrete spectrum entirely and creating a continuous spectrum instead.

9. The barotropic instability

The dispersion function (6.3) can be exploited to study the problem of instability itself. For that we must first derive the continuum eigenfunctions of (5.1) as boundary values on the real c-axis. Instead of making the ansatz (6.1), we now look for complex eigenfunctions of the form

$$\hat{s}(y;c_\mathrm{r},k) = D(c_\mathrm{r},k)\delta\{U(y)-c_\mathrm{r}\} - \frac{S'(y)}{U(y)-(c_\mathrm{r}+\mathrm{i}0)}\hat{\psi}(y;c_\mathrm{r},k). \tag{9.1}$$

Following the same procedure as previously, we now find the dispersion function $D(c_\mathrm{r},k)/|U'(y_\mathrm{c})|$ to be specified by

$$\begin{aligned}\frac{D(c_\mathrm{r},k)}{|U'(y_\mathrm{c})|} &= -k^2 + \int_{y_1}^{y_2} \frac{S'(y)}{U(y)-(c_\mathrm{r}+\mathrm{i}0)}\hat{\psi}(y;c_\mathrm{r},k)\,\mathrm{d}y \\ &= -k^2 + P\int_{y_1}^{y_2} \frac{S'(y)}{U(y)-c_\mathrm{r}}\hat{\psi}(y;c_\mathrm{r},k)\,\mathrm{d}y + \pi\mathrm{i}\frac{S'(y_\mathrm{c})}{U'(y_\mathrm{c})}\hat{\psi}(y_\mathrm{c};c_\mathrm{r},k).\end{aligned} \tag{9.2}$$

The dispersion function thus found can now be employed to investigate whether or not the flow is stable. To that end we introduce the function

$$Z(c,k) = -\frac{D(c,k)}{k^2|U'(y_c)|} = 1 - \frac{1}{k^2}\int_{y_1}^{y_2} \frac{S'(y)}{U(y)-c}\hat{\psi}(y;c,k)\,\mathrm{d}y. \tag{9.3}$$

To determine whether the flow is stable or not one could use the Nyquist method in that one draws the curve C_Z in the complex Z-plane, found by mapping the curve C_c, which encircles the upper half complex c-plane. If C_Z does encircle the origin (counterclockwise) one or more times, the flow is unstable. To find necessary and sufficient conditions for instability in this way depends very much on the kind of profile one considers as well as whether the flow is bounded or unbounded (see Tung 1981). If we take $\hat{\psi}$ in (9.3) to be bounded and positive for all c, then for $|c| \to \infty, Z \to 1$. If furthermore we presume a basic-flow profile $U(y)$ of which the derivative of the absolute vorticity $S' = \beta - U''(y)$ possesses only two zeroes (inflection points), then Z crosses the real Z-axis at only two places, that is where $S'(y_c) = 0$. In this situation it seems that the only way for Z to encircle the origin arises if, and only if,

$$1 - \frac{1}{k^2}P\int_{y_1}^{y_2} \frac{S'(y)}{U(y)-c_{r_1}}\hat{\psi}(y;c_{r_1},k)\,\mathrm{d}y < 0 < 1 - \frac{1}{k^2}P\int_{y_1}^{y_2} \frac{S'(y)}{U(y)-c_{r_2}}\hat{\psi}(y;c_{r_2},k)\,\mathrm{d}y \tag{9.4}$$

where

$$\begin{aligned} c_{r_1} &= U(y_{\mathrm{I}}) \\ c_{r_2} &= U(y_{\mathrm{II}}) \end{aligned} \tag{9.5}$$

in which $S(y)$ attains a local maximum value for $y = y_{\mathrm{I}}$ and a local minimum value for $y = y_{\mathrm{II}}(S'(y)$ goes from positive to negative in $y = y_{\mathrm{I}}$ and from negative to positive in $y = y_{\mathrm{II}}$ and we assume $U'(y = y_{\mathrm{I}}, y_{\mathrm{II}}) \neq 0)$.The first part of (9.4) is satisfied for certain k if

$$P\int_{y_1}^{y_2} \frac{S'(y)}{U(y)-c_{r_1}}\hat{\psi}(y;c_{r_1},k)\,\mathrm{d}y > 0 \tag{9.6a}$$

which is very reminiscent of the Fjørtoft theorem (Fjørtoft 1950, Tung 1981) specialized for the neutral solution and the second part of (9.4) yields

$$P\int_{y_1}^{y_2} \frac{S'(y)}{U(y)-c_{r_2}}\hat{\psi}(y;c_{r_2},k)\,\mathrm{d}y < k^2 \tag{9.6b}$$

where $\hat{\psi}$ is found via (7.8) in terms of the neutral solutions for $c = c_{r1}$ and $c = c_{r2}$. In general we may conclude that, irrespective of the type of basic-flow profile one considers, one thing is clear from (9.3). In order to encircle the origin, Z has to cross the real Z-axis somewhere implying that $S'(y_c) = 0$ (see (9.2)), which results in the Rayleigh–Kuo equation becoming the equation for the neutral solutions. Thus we find that $\hat{\psi}$ in (9.3) is given by (7.8) where $\hat{\psi}_1$ and $\hat{\psi}_2$ are the two neutral solutions of the Rayleigh–Kuo equation.

Instead of (9.2) we also could split Z beforehand into its real and imaginary parts. If we consider only slightly unstable disturbances with $|c_i| \ll c_r$, then

$$0 = k^2 Z(c,k) = k^2 Z_r(c,k) + ik^2 Z_i(c,k) \approx k^2 Z_r(c_r,k) + ik^2 Z_i(c_r,k) + ik^2 c_i \left.\frac{\partial Z_r}{\partial c}\right|_{c=c_r} \tag{9.7}$$

where we have Taylor expanded Z. The term $c_i \partial Z_i/\partial c$ is ignored because it is the product of c_i which is small, and $\partial Z_i/\partial c$, which can be assumed to be small because $c_i \approx Z_i$, which can be seen by equating the real and imaginary parts of (9.7) separately to zero. From (9.7) we thus deduce

$$k^2 Z_r(c_r,k) = k^2 - P\int_{y_1}^{y_2} \frac{S'(y)}{U(y)-c_r}\hat{\psi}(y;c_r,k)\,\mathrm{d}y = 0. \tag{9.8}$$

For the purpose of this integration and to be consistent, we can take $S'(y_c) = 0$, which makes the principal-value sign superfluous. In that case, however, the function $\hat{\psi}$ in (9.8) is in fact nothing else but the neutral solution of the Rayleigh–Kuo equation. Equating the imaginary part of (9.7) to zero yields

$$c_i = \frac{Z_i(c_r,k)}{\partial Z_r/\partial c|_{c=c_r}}. \tag{9.9}$$

Using (9.2) this becomes

$$c_i = \pi\frac{S'(y_c)}{U'(y_c)}\hat{\psi}_{\text{neut}}(y_c;c_r,k)\left\{\left.\frac{\partial(k^2 Z_r)}{\partial c}\right|_{c=c_r}\right\}^{-1} \tag{9.10}$$

where the subscript 'neut' denotes the neutral solution. In the special case that $\hat{\psi}_{\text{neut}}$ does not depend upon c and k (as for example for the Bickley jet $U(y) = \text{sech}^2(y)$, see Lipps (1962)), a further reduction of (9.10) is possible. Consider

$$f\{c_r(k),k\} = k^2 Z_r(c_r,k) = k^2 - \int_{y_1}^{y_2} \frac{S'(y)}{U(y)-c_r}\hat{\psi}_n(y)\,\mathrm{d}y = 0 \tag{9.11}$$

where $c_r(k)$ is defined by $f(c_r,k) = 0$. Differentiating f gives

$$\frac{\mathrm{d}c_r}{\mathrm{d}k}\frac{\partial f}{\partial c_r} + \frac{\partial f}{\partial k} = 0. \tag{9.12}$$

Or with (9.11)

$$\frac{\mathrm{d}f}{\mathrm{d}k} = 2k + \frac{\mathrm{d}c_r}{\mathrm{d}k}\frac{\partial f}{\partial c_r}. \tag{9.13}$$

Combining (9.12) and (9.13) yields

$$\frac{\partial f}{\partial c_r} = -k\left(\frac{\mathrm{d}c_r}{\mathrm{d}k}\right)^{-1}. \tag{9.14}$$

Thus (9.10) can be rewritten into

$$c_i = -\frac{\pi}{k}\frac{\mathrm{d}c_r}{\mathrm{d}k}\frac{S'(y_c)}{U'(y_c)}\hat{\psi}_n(y_c) \tag{9.15}$$

indicating that the neutral solution, the amplitude of which does not depend upon frequency and wavelength, should be dispersive in order to possess neighbouring unstable waves.

10. Solving the integro-differential equation; a long-wave approximation

One of the main difficulties in constructing exact solutions of the integro-differential equation (3.3) is the fact that the kernel $Q(y \mid \eta)$ in (6.4) is non-degenerate. However, for the bounded-flow case we may expand the kernel Q in the complete set of eigenfunctions of the differential operator $\mathrm{d}^2/\mathrm{d}y^2$ in (3.4). Moreover it is then possible to indicate a way to construct a long-wave approximation to the exact dispersion function that contrary to the long-wave approximation of Howard and Drazin (1964) retains the multi-valuedness due to the logarithmic singularity.

The normalized eigenfunctions of $\mathrm{d}^2/\mathrm{d}y^2$ for $0 \leq y \leq l$ are

$$e_n(y) = \sqrt{\frac{2}{l}} \sin\left(n\pi \frac{y}{l}\right) \tag{10.1}$$

where $n = 1, 2, 3, \cdots$, and

$$\lambda_n = -\left(\frac{n\pi}{l}\right)^2 \tag{10.2}$$

are the corresponding eigenvalues. With the general formula for the eigenfunction expansion of a Green function we obtain

$$Q(y \mid \eta) = \sum_{n=1}^{\infty} Q_n(\alpha^2) f_n(y) f_n(\eta) \tag{10.3}$$

where

$$Q_n(\alpha^2) = \frac{2}{l} \alpha^2 \frac{1}{n^2 + \alpha^2} \qquad \alpha = \frac{kl}{\pi} \tag{10.4}$$

and

$$f_n(y) = \sin\left(n\pi \frac{y}{l}\right). \tag{10.5}$$

Substitution of (10.3) in the Fredholm equation (6.4) results in

$$\hat{\phi}(y; c, k) = Q(y \mid y_c) + \frac{1}{k^2} \sum_{n=1}^{\infty} Q_n(\alpha^2) f_n(y) \int_0^l \tilde{f}_n(\eta; y_c) \hat{\phi}(\eta; c, k) \, \mathrm{d}\eta \tag{10.6}$$

where

$$\tilde{f}_n(\eta; y_c) = \frac{f_n(\eta) - f_n(y_c)}{U(\eta) - U(y_c)} S'(\eta) \tag{10.7}$$

and of which the solution is found to be given by

$$\hat{\phi}(y; c, k) = \sum_{n=1}^{\infty} b_n(c, k) f_n(y) \tag{10.8}$$

where

$$b_n(c,k) = Q_n(\alpha^2)\left\{ f_n(y_c) + \frac{1}{k^2}\int_0^l \tilde{F}_n(y;y_c)\hat{\phi}(y;c,k)\,\mathrm{d}y \right\}. \tag{10.9}$$

These coefficients b_n are found in the usual way (Hildebrand 1965).

The dispersion function is now readily obtained from (9.3) and with (10.8) found to be given by

$$Z(c,k) = 1 - \frac{1}{k^2}\sum_{n=1}^{\infty} b_n(c,k) P\int_0^l \frac{S'(y)}{U(y)-c}\sin\left(n\pi\frac{y}{l}\right)\,\mathrm{d}y = 0. \tag{10.10}$$

$Z(c,k)$ contains terms of the form $\ln\{(y_c - l)/y_c\} = \ln[\{U^{-1}(c) - l\}/U^{-1}(c)]$ resulting in multi-valuedness of Z except when $S'(y_c) = 0$ or $\sin(n\pi y_c/l) = 0$ where the latter can only be fulfilled if $y_c = 0$ or $y_c = l$.

Truncating the series in (10.10) results in a long-wave approximation of the dispersion function provided (10.8) is convergent. For that we refer to Hildebrand (1965).

11. The exact dispersion relationship; an alternative to the Fredholm determinant

Instead of seeking approximate solutions of the non-singular integral equation (6.4), one could also start from the original singular integral equation (5.1).

Consider the following expansion of the kernel $Q(y \mid \eta)$ in a complete, but now not necessarily orthogonal, set of functions on the interval (y_1, y_2), say $q_n(y)$, according to

$$Q(y \mid \eta) = \sum_n r_n(\eta) q_n(y) \tag{11.1}$$

where the coefficients r_n will be functions of η. For that we presume that either we are provided with two biorthogonal sets of functions, of which $q_n(y)$ is one, or that the set $q_n(y)$ is a so-called Riesz base (that is $q_n(y) = \mathcal{A}\phi_n(y)$, where $\mathcal{A}$ is a bounded, linear and invertable operator and $\phi_n(y)$ is a orthonormal base). Substitution of (11.1) in (5.1) yields

$$\{U(y) - c\}\hat{s}(y;c,k) = \frac{1}{k^2}\frac{\mathrm{d}S}{\mathrm{d}y}\sum_n s_n(c,k) q_n(y) \tag{11.2}$$

where

$$s_m(c,k) = \int_{y_1}^{y_2} r_m(\eta)\hat{s}(\eta;c,k)\,\mathrm{d}\eta. \tag{11.3}$$

Next we again employ the ansatz (6.1), that is

$$\hat{s}(y;c,k) = D(c,k)\delta\{U(y) - c\} + \frac{1}{k^2}\sum_n s_n(c,k) P\frac{S'(y)}{U(y)-c} q_n(y) \tag{11.4}$$

where D is determined by the normalization (11.3), that is

$$s_m(c,k) = \frac{D(c,k)}{|U'(y_c)|} r_m(y_c) + \frac{1}{k^2} \sum_n s_n(c,k) P \int_{y_1}^{y_2} \frac{S'(y)}{U(y)-c} r_m(y) q_n(y)\, \mathrm{d}y \tag{11.5}$$

or

$$\begin{aligned} \frac{D(c,k)}{|U'(y_c)|} r_m(y_c) &= s_m(c,k) - \frac{1}{k^2} \sum_n s_n(c,k) p_{nm}(c,k) \\ &= \frac{1}{k^2} \sum_n \{k^2 \delta_{nm} - p_{nm}\} s_n(c,k) \end{aligned} \tag{11.6}$$

where

$$p_{nm}(c,k) = P \int_{y_1}^{y_2} \frac{S'(y)}{U(y)-c} q_n(y) r_m(y)\, \mathrm{d}y. \tag{11.7}$$

Setting D equal to zero in (11.6), the resulting set of linear homogeneous equations for s_n yield non-zero solutions only if the determinant of the coefficients vanishes:

$$\begin{vmatrix} k^2 - p_{00} & -p_{10} & -p_{20} & \cdots \\ -p_{01} & k^2 - p_{11} & -p_{21} & \cdots \\ -p_{02} & -p_{12} & k^2 - p_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 0 \tag{11.8}$$

or, more succinctly,

$$|k^2 \delta_{nm} - p_{nm}(c,k)| = 0. \tag{11.9}$$

Provided that this determinantal equation is convergent it may, for given k, be solved for the possible values of c and can be viewed upon as an alternative to the Fredholm determinant for (6.14).

A special situation arises when the functions $q_n(y)$ and $r_m(y)$ are orthogonal for $n \neq m$ in the scalar product

$$\langle f, g \rangle = \int_{y_1}^{y_2} W(y;c) f^*(y) g(y)\, \mathrm{d}y \tag{11.10}$$

where the weight function W is given by

$$W(y;c) = \frac{S'(y)}{U(y)-c} \neq 0. \tag{11.11}$$

In this case p_{nm} becomes diagonal and (11.9) reduces to

$$|\{k^2 - h_n(c,k)\} \delta_{nm}| = 0 \tag{11.12}$$

where

$$h_n(c,k) = \int_{y_1}^{y_2} W(y;c) q_n(y) r_n(y)\, \mathrm{d}y. \tag{11.13}$$

The eigenvalues are then $c = c_n$, where $h_n(c,k) = k^2$. Basic-flow profiles $U(y)$ that give rise to a diagonal p_{nm} should satisfy the differential equation

$$\frac{\mathrm{d}^2 U}{\mathrm{d}y^2} + W(y;c) U = \beta + cW(y;c) \tag{11.14}$$

where we choose the c-dependence of W (or alternatively β) such that U does not depend upon c.

12. Summary and conclusions

In the present paper we have tried to establish an integral-equation approach to the (linear) barotropic instability of a sheared zonal flow. Thus we were able to derive some results that appear to be new.

Starting from an integro-differential equation with a non-degenerate kernel for the vorticity perturbation we present a formal solution of the barotropic-instability problem. A normal-mode analysis of our integro-differential equation yields singular eigenfunctions that have the same structure as the van Kampen modes in a hot plasma. By classifying these eigenfunctions we identify a new expression for the dispersion relationship governing the instability. Subsequently we use this form of the dispersion relationship to derive expressions for the growth rate of a marginally instable mode in terms of the flow field and the neutral solutions. Furthermore we have discussed the relevance of the continuous spectrum associated with the instability eigenvalue problem thereby identifying that contrary to what is stated in the literature, the dispersion function is multi-valued. The disappearance of this continuous spectrum, for example for a vortex sheet, is conjectured to be a manifestation of the mathematical phenomenon of spectral concentration. Finally we give an exact explicit expression for the dispersion relationship in the form of a determinantal equation that involves only the basic-flow profile and the phase speed. We discuss some special cases in which this determinantal equation, that is in fact an alternative for the Fredholm determinant, can be solved exactly.

As we have hoped to demonstrate, the integral-equation formulation of the barotropic-instability problem has some conclusive advantages over the conventional differential-equation approach via the Rayleigh–Kuo equation. In particular we have shown that within this framework the normal-mode analysis is simplified from both the theoretical and practical point of view. In essence this is due to the fact that the formulation enables the isolation, or separation, of the continuous spectrum and a subsequent reduction of the problem to a much simpler one that is free of inconvenient singularities and is tractable by classical methods.

As we have pointed out in section 6 the normal modes (6.1), although singular and not governed by any dispersion law, enable the construction of physically and mathematically acceptable perturbations by the superposition of many such modes. Thus it is possible to solve the initial-value problem for the barotropic instability in a way that is, in principle, identical to the Laplace-transform method. In this picture the instability can be understood as the constructive interference of many singular eigenmodes of the form (6.1). This phase mixing in physical space results in an algebraically damped contribution due to the continuous spectrum whereas the dispersion law for the discrete spectrum governs the exponentially damped or amplified collective modes.

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